AN ALTERNATIVE APPROACH TO ASYMPTOTIC EXPANSION OF THE DISTRIBUTION OF THE LIKELIHOOD-RATIO CRITERION

By

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1. INTRODUCTION

Let $W (0 \leqslant W \leqslant 1)$ be a statistic whose h^{th} order moment is as under

$$E(W^h) = K \left(\frac{\prod_{j=1}^{b} y_j^{y_j}}{\prod_{k=1}^{a} x_k^{x_k}} \right)^h \frac{\prod_{k=1}^{a} \Gamma\{x_k(1+h)+\xi_k\}}{\prod_{j=1}^{b} \Gamma\{y_j(1+h)+\eta_j\}} \tag{1.1}$$

where K is a constant such that $E(W^o)=1$ and

$$\sum_{k=1}^{a} x_k = \sum_{j=1}^{b} y_j \tag{1.2}$$

Box [3] has given a general asymptotic expansion of the distribution of W. The same expression has also been reproduced by Anderson in his book [1]. Using the moments Box has first determined the characteristic function of the statistic. He has then applied expansion formula for the gamma function due to Barnes [2] and then using the inversion theorem for characteristic function he has obtained the density function of the statistic.

In this paper, we obtain the asymptotic density function of the same statistic by a simple method. We avoid the use of characteristic function and directly apply expansion formula (2.3a) to the h^{th} order moment, μ_h, of the statistic and then obtain the density function by simply taking the inverse Mellin transform of μ_h. In the process Box has introduced a dummy multiplier ρ $(0 \leqslant \rho < 1)$ to the statistic involved. But we have avoided introducing ρ and have gone ahead without it. This considerably simplified our procedure.

2. Some Preliminary Results

Some important results and formulae employed in what follows are as under :

(*i*) Consider, for a complex variable s, the functions $f(x)$ and $F(s)$ related as follows :

$$F(s)=\int_0^\infty f(x)\, x^{s-1}\, dx$$

where $F(s)$ is called the Mellin transform of $f(x)$. The inverse Mellin transform $f(x)$ of $F(s)$ is defined [5] as follows :

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s}\, ds \qquad \text{.. (2.1)}$$

(*ii*) An important result [4] which will be used in the sequel is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\nu)\,(s+a)^{-\nu}\, x^{-s}\, ds=\begin{cases} x^{a}\,(-\log x)^{\nu-1} & 0<x\leqslant 1 \\ 0 & 1<x<\infty \end{cases}$$

$$\text{Re}\ \nu>0,\ \text{Re}\ s>\text{Re}\ a \qquad \text{...(2.2)}$$

(*iii*) An expansion formula for the gamma function [2] which is asymptotic in x for bounded h is

$$\Gamma(x+h)=\sqrt{2\pi}\ x^{x+h-\frac{1}{2}}\ \exp\left\{-x-\sum_{r=1}^{m}\frac{(-1)^r}{r(r+1)}\frac{B_{r+1}{}^{(h)}}{x^r}+R_{m+1}(x)\right\} \qquad \text{...(2.3a)}$$

where $R_{m+1}(x)=0(x^{-(m+1)})$ and $B_r^{(h)}$ is the Bernoulli polynomial of degree r and order unity defined by

$$\frac{\tau e^{h\tau}}{e^{\tau}-1}=\sum_{r=0}^{\infty}\frac{\tau^r}{r!}B_r^{(h)} \qquad \text{...(2.3b)}$$

3. Asymptotic Distribution

Changing $h=s-1$ in (1.1) and using (2.3a), we get

$$E(W^{(s-1)})=s^{\left\{\sum\limits_{k=1}^{a}\xi_k-\sum\limits_{j=1}^{b}\eta_j-\frac{1}{2}(a-b)\right\}}\exp\Big[-\sum_{j=1}^{b}\sum_{r=1}^{m}\frac{(-1)^r}{r(r+1)}\frac{B_{r+1}(\eta_j)}{y_j^r}$$

$$\left(1-\frac{1}{s^r}\right)+\sum_{k=1}^{a}\sum_{r=1}^{m}\frac{(-1)^r}{r(r+1)}\frac{B_{r+1}(\xi_k)}{x_k^r}\left(1-\frac{1}{s^r}\right)+\sum_{k}0(x_k^{-(m+1)})$$

$$+\sum_{j}0(y_j^{-(m+1)})\Big]$$

$$=s^{-f/2}\exp\left[\sum_{r=1}^{m}\left(\frac{1}{s^r}-1\right)\omega_r-R'_{m+1}\right] \qquad \ldots(3.1)$$

where $f=-2\left[\sum\limits_{k=1}^{a}\xi_k-\sum\limits_{j=1}^{b}\eta_j-(a-b)/2\right]$

$$\omega_r=\frac{(-1)^{r+1}}{r(r+1)}\left\{\sum_{k=1}^{a}\frac{B_{r+1}(\xi_k)}{x_k^r}-\sum_{j=1}^{b}\frac{B_{r+1}(\eta_j)}{y_j^r}\right\} \qquad \ldots(3.1a)$$

and $R'_{m+1}=\sum\limits_{k}0\,(x_k^{-(m+1)})+\sum\limits_{j}0\,(y_j^{-(m+1)})$

Using the power series expansion of e^x we rewrite (3.1) as

$$E(W^{s-1})=s^{-f/2}\left[1+\left(\frac{1}{s}-1\right)\omega_1+\left(\frac{1}{s^2}-1\right)\omega_2+\left(\frac{1}{s^2}-\frac{2}{s}+1\right)\right.$$

$$\left.\frac{\omega_1^2}{2}+\ldots+R''_{m+1}\right] \qquad \ldots(3.2)$$

Now taking the inverse Mellin transform of $E(W^{s-1})$, the asymptotic density $f(W)$ of W is given as below :

$$f(W)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}W^{-s}.s^{-f/2}\left[1+\left(\frac{1}{s}-1\right)\omega_1+\left(\frac{1}{s^2}-1\right)\right.$$

$$\omega_2+\left(\frac{1}{s^2}-\frac{2}{s}+1\right)\frac{\omega_1^2}{2}+\,..+R''_{m+1}\Big]\,ds$$

Using (2.2) we get

$$f(W)=\frac{(-\log W)^{f/2-1}}{\Gamma(f/2)}+\omega_1\left\{\frac{(-\log W)^{(f+2)/2-1}}{\Gamma[(f+2)/2]}-\frac{(-\log W)^{f/2-1}}{\Gamma(f/2)}\right\}$$

$$+\omega_2\left\{\frac{(-\log W)^{(f+4)/2-1}}{\Gamma[(f+4)/2]}-\frac{(-\log W)^{f/2-1}}{\Gamma(f/2)}\right\}$$

$$+\frac{\omega_1^2}{2}\left\{\frac{(-\log W)^{(f+4)/2-1}}{\Gamma[(f+4)/2]}-2\,\frac{(-\log W)^{(f+2)/2-1}}{\Gamma[(f+2)/2]}\right.$$

$$\left.+\frac{(-\log W)^{f/2-1}}{\Gamma(f/2)}\right\}+\ldots+R'''_{m+1}; \qquad (3.3)$$

Setting $-2\log W=Z$ and integrating (3.3) from 0 to z_0, we get the asymptotic distribution funciion of Z as follows :-

$$P_r\,(Z\leqslant z_0)=P_r\,(\chi_f^2\leqslant z_0)+\omega_1\,\{P_r\,(\chi^2_{f+2}\leqslant z_0)-P_r(\chi_f^2\leqslant z_o)\}$$

$$+[\omega_2\,\{P_r\,(\chi^2_{f+4}\leqslant z_0)-P_r\,(\chi_f^2\leqslant z_0)\}$$

$$+\frac{\omega_1{}^2}{2}\{P_r\,(\chi^2_{f+4}\leqslant z_0)-2\,P_r\,(\chi^2_{f+2}\leqslant z_0)$$

$$+P_r\,(\chi_f^2+\leqslant z_0)\}]+\ldots\;\ldots..+R_{m+1} \qquad ...(3.4)$$

where χ_f^2 is chi-square with f degrees of freedom.

It will be clear that the abovep rocedure is simpler than that given by Box [3]. Further, the result (3.4) agrees with that of Box [3] or Anderson [1], the only exception being that in his result there is a ρ-multiplier for which different values have got to be inserted for finding the distributions of different test statistics. It is this factor ρ which has also made his results more complicated.

4. Illustration

(*i*) Consider for instance (Anderson 1958, pp. 191) the *L-R* criterion, λ, for testing the general linear hypothesis about regression coefficients. By setting in (3.4) the following :-

$$W=\lambda,\ a=b=p,\ x_k=\frac{N}{2}\ \ y_j=\frac{N}{2},\ \zeta_k=\frac{1}{2}(-q+1-k)$$

$$\eta_j=\frac{1}{2}(-q_2+1-j),\ i,j=1,2,\ldots,p$$

we obtain

$$P_r(-2\log\lambda\leqslant z_0)=P_r(\chi^2_{pq1}\leqslant z_0)+\frac{pq_1}{4N}(p+1+q_1+2q_2)$$

$$\{P_r(\chi^2_{pq1+2}\leqslant z_0)$$

$$-P_r(\chi^2_{pq1}\leqslant z_0)\}+\frac{pq_1}{12N^2}\{\frac{(p+1)(2p+1)}{2}+\frac{3}{2}$$

$$(p+1)(q_1+2q_2)+(q_1^2+3q_2^2+3q_1q_2-\frac{2}{3})\}$$

$$\{P_r(\chi^2_{pq1}+4\leqslant z_0)-P_r(\chi^2_{pq1}\leqslant z_0)\}+\frac{p^2q_1^2}{32N^2}(p+1+q_1+2p_2)^2$$

$$\{P_r(\chi^2_{pq1+4}\leqslant z_0)-2P_r(\chi^2_{pq1+2}\leqslant z_0)+P_r(\chi^2_{pq1}\leqslant z_0)\}+\ldots \quad \ldots(3.5)$$

(*ii*) Consider next (Anderson 1958, pp. 238) the *L-R* criterion, λ, for testing independence of q sets of variates. Setting

$$W=\lambda,\ a=b=p,\ x_k=\frac{N}{2},\ \xi_k=-\frac{k}{2},\ k=1,2,\ldots,p\,;$$

$$y_j=\frac{N}{2},\ \eta_j=\frac{-j+p_1+\ldots+p_{i-1}}{2},\ j=p_1+p_{i-1}+1,\ldots,$$

$$p_1+p_2+p_3+\ldots+p_i\,;\ i=1,2,\ldots,q,$$

we obtain

$$P_r(-2\log z_0\leqslant z_0)=P_r\chi^2_{(p^2-\Sigma p_1^2}\leqslant z_0)+\frac{1}{24N}\{2(p^3-\Sigma p_i^3$$

$$+9(p^2-\Sigma p_i^2)\}\times\{P_r(\chi^2_{(p^2-\Sigma p_i^2})/2+2\leqslant z_0)-P_r(\chi^2_{(p^2-\Sigma p_i^2)}/2\leqslant z_0)\}$$

$$+\frac{1}{48N^2}\{(p^4-\Sigma p_i^4)+6(p^3-\Sigma p_i^3)+11(p^2-\Sigma p_i^2)\}$$

$$\times\{P_r(\chi^2_{(p^2-\Sigma p_1^2)/2+4}\leqslant z_0)-P_r(\chi^2_{(p^2-\Sigma p_i^2)}/2\leqslant z_0)\}$$

$$+\frac{1}{2\,(24)^2N^2}\,\{2\,(p^3-\Sigma\, p_i^3)+9\,(p^2-\Sigma\, p_i^2)\}^2$$

$$\times\{P_r\,(\chi^2_{(p2-\Sigma p_i^2)}/2+4\leqslant z_0)-2P_r(\chi^2_{(p2-\Sigma p_i^2)}/2+2\leqslant z_0)$$

$$+P_r\,(\chi^2_{(p2-\Sigma p_i^2)}/2\leqslant z_0)\}+\ldots \qquad \ldots(3.6)$$

Similar expressions for the other criteria such as for testing the equality of covariance matrices, testing that q normal populations are identical, sphericity test etc, can also be obtained by making suitable substitutions in (3.4),

References

[1] Anderson, T.W., (1958) : *An Introduction to Multivariate Statistical Analysis.* John Wiley, New York.

[2] Barnes, E.W. (1899) : "The theory of the gamma function", *Messeng. Math.*, **29**, 64-128.

[3] Box, G.E.P. (1949) : "A general distribution theory for a class of likelihood criteria." *Biometrika*, **36**, 317-346.

[4] Erdélyi, (1954) : *Tables of Integral Transforms* Vol. 1, McGraw Hill, New York.

[5] Titchmarsh, E.C., (1948) : *Introduction to Theory of Fourier Integrals.* University Press, Oxford.

[6] Whittaker, E.T., and Watson, G.N. (1943) : *A course of Modern Analysis*, Cambridge University Press and Macmillan Co.